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Horn, Peter Harry

University of California

THE TRANSMISSIBILITY OF MACHINE MOUNTINGS HAVING NON-LINEAR SPRING CHARACTERISTICS

PETER HARRY HORN

The Transmissibility of Machine Mountings Having

Non-Linear Spring Characteristics

By

Peter Harry Horn, 1908-

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Approved:

/s/ C. F. Garland

/s/ L. M. K. Boelter

/s/ ~~L. B. Loeb~~ Changed to B. M. Woods P.H.H.

Committee in Charge

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INTRODUCTION

The Transmissibility of any vibrating system can be discussed only after the displacement amplitude of vibration is determined. The displacement amplitude, in turn, is usually expressed in the form of a differential equation. Therefore, it is necessary to discuss briefly differential equations and their solutions.

In the solution of elementary vibration problems, certain assumptions are usually made regarding the damping force and the spring force of the vibrating system. The damping force is assumed to be a linear function of the velocity of motion. The restoring force of the spring at any instant is assumed to be proportional to the deformation. With these assumptions the equation of motion is a linear differential equation with constant coefficients. The solution to a linear differential equation is easily obtained and the displacement amplitude of the vibration determined. From the solution of the equation of motion, an expression is then derived for the Transmissibility of the system.

The spring force may not be a linear function of the displacement if rubber, leather, cork or plastics are used. The linear differential equation with constant coefficients is no longer sufficient to describe the motion of systems employing the above-mentioned materials. A general investigation of such systems requires a discussion

of non-linear differential equations, and for the particular case at issue, a discussion of a "system with a non-linear spring characteristic."

The general solution of a differential equation for the forced vibrations of a system with a non-linear spring characteristic is unknown. The method of superposition of vibrations which is always applicable in the case of linear systems is no longer valid. If the free vibrations and the forced vibrations of the system are found, the sum of the two motions does not give the resultant vibration. To simplify the problem, this paper will discuss only the steady forced vibrations, and neglect the discussion of the free vibrations that depend on the initial condition.

The solution to the equation of motion for a system with a non-linear spring, presented in this paper, yields a close approximation to the displacement amplitudes found by experiment, and is of such a form that the Transmissibility can be readily discussed. For this reason it has decided advantages over the present methods of solution, i.e.,

1. Numerical Integration.
2. Graphical Integration.
3. Method of Successive Approximations.
4. Graphical solutions neglecting damping. (Ref. b-page 139)

TABLE OF NOMENCLATURE

Symbol	Description	Dimension
a	Displacement Amplitude of Vibration	L
b	Coefficient of Viscous Damping	$M\ T^{-1}$
c	$\frac{3}{4}\frac{K_1}{K}$ Ratio of Spring Constants	L^{-2}
e	Eccentricity of Unbalanced Mass	L
F	Spring Force	$M\ L\ T^{-2}$
k	Linear Spring Constant	$M\ T^{-2}$
k_1	Non-linear Spring Constant	$M\ L^{-2}T^{-2}$
M	Total Mass in Vibration	M
m	Unbalanced Mass	M
2n	$\frac{b}{M}$	T^{-1}
p	$\frac{me}{M}$	L
Q	$me\,\omega_f^2$ Amplitude of Exciting Force	$M\ L\ T^{-2}$
y	Displacement in y direction	L
$\dot{y}$	$\frac{dy}{dt}$ Velocity in y direction	$L\ T^{-1}$
$\ddot{y}$	$\frac{d^2y}{dt^2}$ Acceleration in y direction	$L\ T^{-2}$
ω_f	Exciting frequency	T^{-1}
ω_N	$\sqrt{\frac{K}{M}}$ Natural frequency	T^{-1}
R	$\frac{\omega_f}{\omega_N}$ FREQUENCY RATIO	Dimensionless
G	$\frac{2n}{\omega_N}$ DAMPING FACTOR	Dimensionless
E	$c\ p^2$ EXCITING FORCE	Dimensionless
A^2	$c\ a^2$ AMPLITUDE FACTOR	Dimensionless
T	Transmissibility	Dimensionless

ABOVE ITEMS ARE IN "CONSISTENT UNITS".

DERIVATION OF FORMULAS

The solution is based on the assumption that the spring characteristic shown in Figure 1, can be represented with reasonable accuracy by an equation of the form

$$F = k\,y + k_1 y^3$$

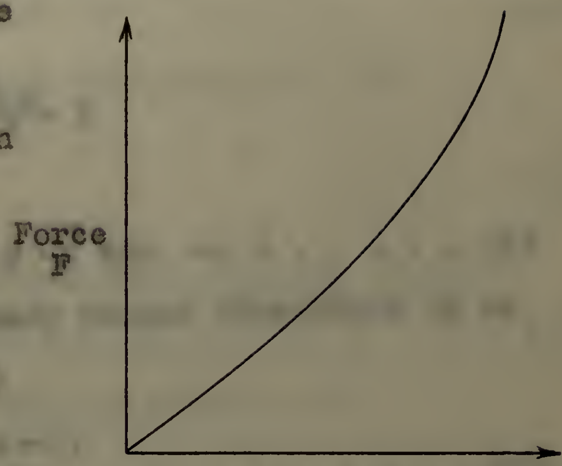


Figure 1.

The second assumption is that the damping is viscous, i.e., the damping force is proportional to the velocity of motion.

The third assumption is that the third power of the sine of an angle is approximately equal to three-fourths the sine of the angle.

Considering the system with one degree of freedom shown in Figure 2, the following forces are involved:

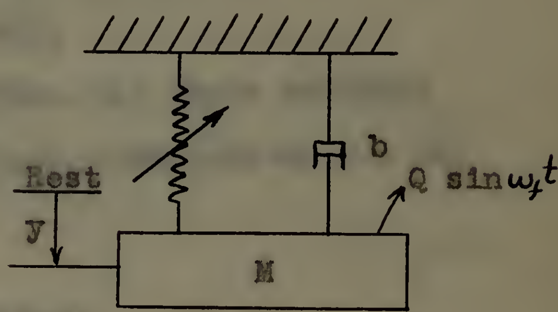


Figure 2.

1. Damping force $= b\,\dot{y}$
2. Spring force $= k\,y + k_1 y^3$
3. Exciting force $= Q \sin \omega_f t \quad (Q = m\,e\,\omega_f^2)$

From Newton's Second Law of Motion

$$Q \sin \omega_f t - b\,\dot{y} - (k\,y + k_1 y^3) = M\,\ddot{y} \;.\;.\;.\;.\;.\;.\;.\;.\;.(1)$$

Rearranging equation (1)

$$M\ddot{y} + b\dot{y} + ky + k_1 y^3 = m e \omega_f^2 \sin \omega_f t \quad . \ . \ . \ . \ . \ . \ . \ (2)$$

Defining

$$\frac{b}{M} = 2n \ ; \quad \frac{k}{M} = \omega_N^2 \quad ; \quad \frac{m e}{M} = p$$

Substituting into equation (2)

$$\ddot{y} + 2n\dot{y} + \omega_N^2 \left(y + \frac{k_1}{k} y^3\right) = p\,\omega_f^2 \sin \omega_f t \quad . \ . \ . \ . \ . \ (3)$$

Assuming the solution for the steady forced vibrations to be

$$y = a \sin(\omega_f t - \alpha)$$

$$\text{Then} \quad \dot{y} = a\,\omega_f \cos(\omega_f t - \alpha)$$

$$\ddot{y} = -a\,\omega_f^2 \sin(\omega_f t - \alpha) \quad . \ . \ . \ . \ . \ . \ (4)$$

$$y^3 = a^3 \sin^3(\omega_f t - \alpha) \cong \tfrac{3}{4} a^3 \sin(\omega_f t - \alpha)$$

The third harmonic of the y^3 term in equation (4) is neglected in this solution. This procedure is followed by E. V. APPLETON (Ref. a), and is discussed in detail by S. TIMOSHENKO (Ref. b - pages38 and 48).

Substituting equations (4) into equation (3) there results;

$$\left[a\omega_N^2 - a\omega_f^2 + \tfrac{3}{4} a\,\omega_N^2 \frac{k_1}{k}\right]\sin(\omega_f t - \alpha) + 2na\,\omega_f \cos(\omega_f t - \alpha) = p\,\omega_f^2 \sin \omega_f t \quad . \ . \ (5)$$

Let $\frac{3}{4}\frac{k_1}{k} = c$

Substituting the following relationships;

$$\sin(\omega_f t - \alpha) = \sin\omega_f t \cos\alpha - \cos\omega_f t \sin\alpha.$$

$$\cos(\omega_f t - \alpha) = \cos\omega_f t \cos\alpha + \sin\omega_f t \sin\alpha.$$

Then, since the coefficients of the sine and cosine terms respectively must be equal,

$$\left[a\omega_N^2 - a\omega_f^2 + ca^3\omega_N^2\right]\cos\alpha\,\sin\omega_f t + 2na\omega_f\sin\alpha\,\sin\omega_f t = p\omega_f^2\sin\omega_f t \qquad (6\text{-a})$$

$$-\left[a\omega_N^2 - a\omega_f^2 + ca^3\omega_N^2\right]\sin\alpha\,\cos\omega_f t + 2na\omega_f\cos\alpha\,\cos\omega_f t = 0 \quad \ldots \quad (6\text{-b})$$

When $\omega_f t = 0$; $\cos\omega_f t = 1.0$ and from equation (6-b)

$$\operatorname{Tan}\alpha = \frac{2n\,\omega_f}{\left[\omega_N^2 - \omega_f^2 + a^2\omega_N^2 c\right]} \quad \ldots\ldots \quad (7)$$

When $\omega_f t = \pi/2$; $\sin\omega_f t = 1.0$ and from equation (6-a)

$$\left[a\omega_N^2 - a\omega_f^2 + a^3\omega_N^2 c\right]\cos\alpha + 2na\,\omega_f\sin\alpha = p\,\omega_f^2 \quad \ldots\ldots \quad (8)$$

Substituting for $\cos\alpha$ and $\sin\alpha$ from equation (7)

$$a^2\left[\left(\omega_N^2 - \omega_f^2 + a^2\omega_N^2 c\right)^2 + 4n^2\omega_f^2\right] = p^2\omega_f^4 \quad \ldots\ldots \quad (9)$$

Thus equation (4) satisfies equation (2) provided the amplitude and phase relations are as specified in equations (7) and (9). For other positions of the vibrating system, equation (2) is not usually satisfied, and the actual motion cannot be represented by the assumed simple harmonic motion.

Assuming the dimensionless groups:

$$c\,a^2 = A^2;\ c\,p^2 = E;\ \frac{2n}{\omega_N} = G;\ \frac{\omega_f}{\omega_N} = R.$$

and substituting into equations (7) and (9),

$$\operatorname{Tan}\alpha = \frac{G\,R}{1 - R^2 + A^2} \quad \ldots\ldots \quad (12)$$

and $$A^2\left[\left(1 - R^2 + A^2\right)^2 + G^2R^2\right] = E\,R^4 \quad \ldots\ldots \quad (14)$$

Equation (14) is the "Amplitude Equation" in dimensionless form. The general shape of the amplitude curve is shown in Figure 3.

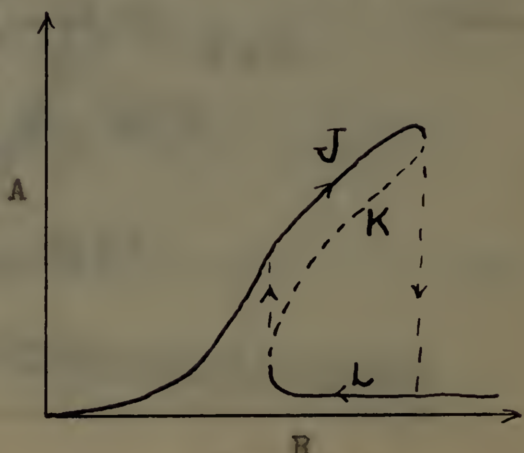


Figure 3.

The branches of the curve marked J and L are the stable amplitudes of the vibration , the branch marked K is unstable and is never realized. The problem of stability is thoroughly discussed by E. V. APPLETON (Ref. a).

When the Amplitude Factor A is plotted against the Frequency Ratio R there are two parameters that may affect the shape of the curves, namely; the Exciting Force E, and the Damping Factor G. The independent effect of these two variables is shown in Figures 4 and 5.

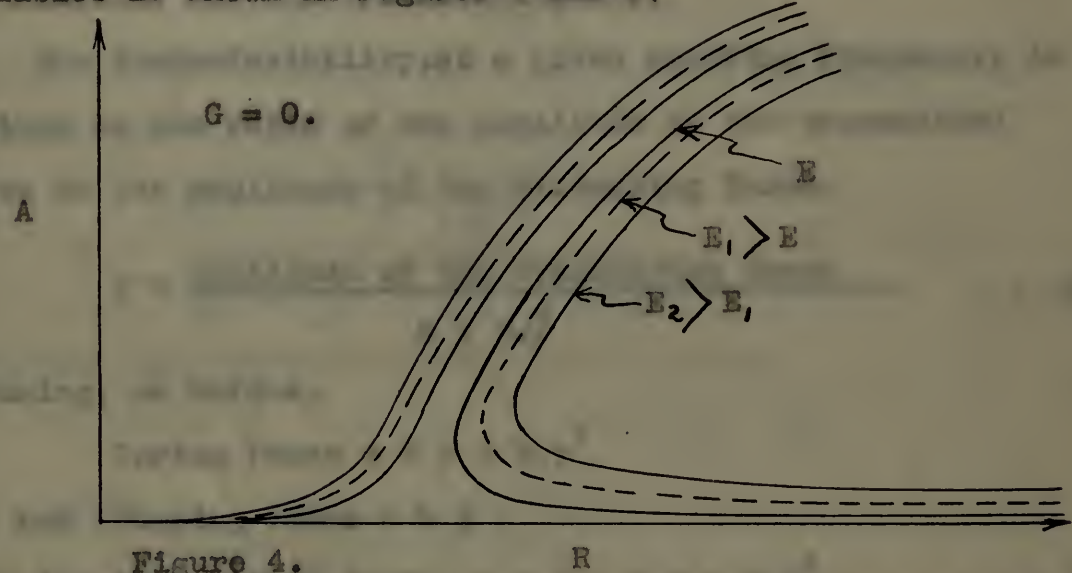


Figure 4.

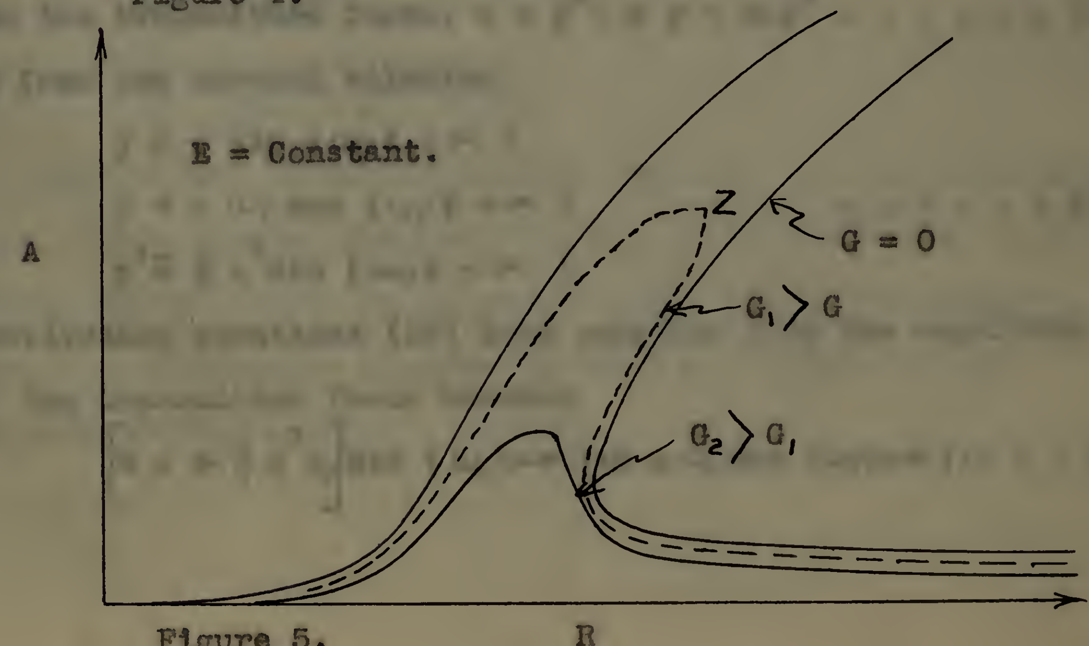


Figure 5.

Referring to Figure 5, it can be seen that the Damping Factor G determines the point marked Z where the amplitude curves round off. For values of $0 \leq G^2 < E$ the curves will never round off, but will extend to infinity without closing. For the case where $G^2 > E$ the curves will round off at definite values of R and A.

With the approximate solution of the differential equation of motion available, it is now possible to derive the equation for the Transmissibility.

The Transmissibility, at a given exciting frequency, is defined as the ratio of the amplitude of the transmitted force to the amplitude of the disturbing force.

$$T = \frac{\text{Amplitude of the Transmitted Force}}{m\,e\,\omega_f^2} \quad . \; . \; . (15)$$

Assuming, as before,

Spring Force $= k\,y + k_1 y^3$

and Damping Force $= b\,\dot{y}$

Then the transmitted force, $= b\,\dot{y} + k\,y + k_1 y^3$ . . . . (16)

But from the assumed solution

$$\begin{aligned} y &= a \sin(\omega_f t - \alpha) \\ \dot{y} &= a\,\omega_f \cos(\omega_f t - \alpha) \\ y^3 &\cong \tfrac{3}{4} a^3 \sin(\omega_f t - \alpha) \end{aligned} \quad . \; . \; . \; . \; . \; . \; . (17)$$

Substituting equations (17) into equation (16) the amplitude of the transmitted force becomes

$$\left[k\,a + \tfrac{3}{4} a^3 k_1\right] \sin(\omega_f t - \alpha) + b\,a\,\omega_f \cos(\omega_f t - \alpha) \ldots \; . \; . (18)$$

or the Amplitude of the Transmitted Force

$$= \sqrt{\left(k\,a + \tfrac{3}{4}\,a^3 k_1\right)^2 + \left(b\,a\,\omega_f\right)^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (19)$$

$$= k\,a\sqrt{\left(1 + c\,a^2\right)^2 + \left(\frac{b\,\omega_f}{k}\right)^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (20)$$

Taking the square root of both sides of equation (9), there results;

$$m\,e\,\omega_f^2 = M\,a\sqrt{\left(\omega_N^2 - \omega_f^2 + a^2\omega_N^2 c\right)^2 + 4\,n^2\omega_f^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (21)$$

Substituting equations (20) and (21) into equation (15)

$$T = \frac{k\,a\sqrt{\left(1 + c\,a^2\right)^2 + \left(\frac{b\,\omega_f}{k}\right)^2}}{M\,a\sqrt{\left(\omega_N^2 - \omega_f^2 + a^2\omega_N^2 c\right)^2 + 4\,n^2\omega_f^2}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (22)$$

Since $\omega_N^2 = \frac{k}{M}$, dividing the numerator and denominator of equation (22) by ω_N^2 there results,

$$T = \frac{\sqrt{\left(1 + c\,a^2\right)^2 + \left(\frac{b\,\omega_f}{k}\right)^2}}{\sqrt{\left(1 - \frac{\omega_f^2}{\omega_N^2} + c\,a^2\right)^2 + \frac{4\,n^2\omega_f^2}{\omega_N^4}}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (23)$$

Recalling the following dimensionless groups

$$A^2 = c\,a^2 \;;\quad R = \frac{\omega_f}{\omega_N} \;;\quad G = \frac{2\,n}{\omega_N} = \frac{b}{M\,\omega_N}$$

and substituting into equation (23)

$$T = \sqrt{\frac{\left(1 + A^2\right)^2 + G^2R^2}{\left(1 - R^2 + A^2\right)^2 + G^2R^2}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (24)$$

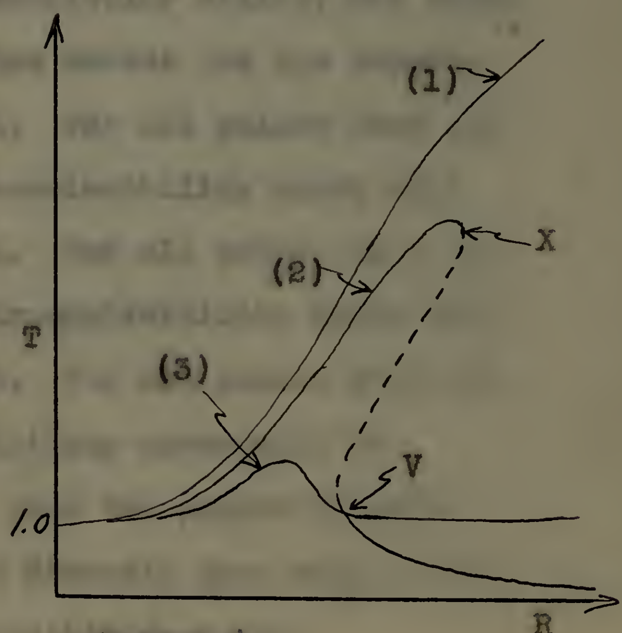
This is the "Transmissibility Equation" in dimensionless form. The variables involved are the Amplitude Factor A, the Damping Factor G and the Frequency Ratio R. The Exciting Force E is indirectly involved through its effect on the Amplitude Factor A (Fig. 4 page 9).

CONCLUSIONS

The data obtained in the laboratory agreed closely with the magnitude of the Amplitude Factors computed from the derived formula (see curves, pages 26 - 29). For the most important branch of the amplitude curve the average error is 3.6 per cent. This result indicates that the derived formula can be used with accuracy, to predict the amplitude of vibration for a system with a non-linear spring; provided the physical constants of the system are available.

The expression for the Transmissibility can be assumed to be correct since it is obtained from the amplitude equation. The Transmissibility curves for a calculated value of Damping Factor G (pages 34 and 38), and for two different values of Exciting Force E are shown on pages 30 - 31.

Referring to sketch A, three types of Transmissibility curves are shown. They are similar in shape to the amplitude curves previously discussed. The lower branches of the Transmissibility curves cross at the point marked V where the Transmissibility is unity and the Frequency Ratio R is slightly greater than $\sqrt{2}$. $\left(R = \sqrt{2(1 + A^2)}\right)$



Sketch A.

When a machine is operating at a constant speed below or near the resonant frequency ($R \leq 1$), the amplitude and Transmissibility are lower than for the same machine with a linear mounting having identical linear constants.

When the operating speed of the machine is greater than the resonant frequency ($R > 1$), a general statement cannot be made regarding the Transmissibility. For the condition where $G^2 < E$, the Transmissibility is relatively large as shown by curve (1) in sketch A. For the condition where $G^2 > E$, the Transmissibility curve (2), sketch A, will be obtained. The point X where the curve rounds off, can be controlled by varying G or E when substituted in the formula $R = \sqrt{\dfrac{G^2}{G^2 - E}}$. For very large values of the Damping Factor G, and low values of Exciting Force E, the Transmissibility is very low as shown by curve (3), sketch A.

Since the values of G and E play an important part in the Transmissibility, curves of particular utility are shown on page 33. The coordinates of the curves are the Damping Factor G and the Exciting Force E. For all points that lie below the curve marked N, the Transmissibility curve will be similar to curve (1), sketch A. For all points that lie between the curves M and N, the Transmissibility curve will be similar to curve (2), sketch A. For all points that lie above the curve M, the Transmissibility curve will be similar to curve (3), sketch A. Thus the proper relative values of E and G can be obtained directly from page 33 to give any desired shape of Transmissibility curve.

The above statements regarding the values of Damping Factor G, and Exciting Force E, should be used to insure that the Transmissibility curve is the same as curve (3), or the lower branch of curve (2), sketch A, when the Frequency Ratio R is greater than unity.

PHOTOGRAPH OF APPARATUS

DESCRIPTION OF APPARATUS

The apparatus used is quite simple. To a reinforced concrete foundation (A) of three feet elevation, two pieces of rolled steel 1/2" x 5", separated by a distance of one foot, are secured so that two cantilever springs (B) are formed. A piece of 6" channel (C) is secured vertically to the ends of the cantilever springs so that vertical motion of the channel piece results when the two parallel cantilever springs deflect. On top of the vertical channel, a 1/4 horse power D. C. motor (D) is firmly secured. The bottom of the vertical channel is connected to the middle of a steel bar (E). The bar (E) is so supported that it can be considered as a beam fixed at both ends and deflected at the middle. The beam supports are in turn secured to a bed-plate buried in reinforced concrete.

To each end of the shaft of the motor, two similar eccentric pieces of steel (F) of known mass and eccentricity are secured.

A rheostat (G) is inserted in the armature circuit of the motor so that a speed range up to 2000 rpm is available. The motor power supply is a 110 volt synchronous converter.

A piece of sheet metal (H) 6" square is rigidly secured to the end of the motor frame. The upper horizontal edge of (H) is squared, smoothed and blackened.

A Strobotac (J), calibrated in rpm, is used to obtain the frequency of rotation of the motor.

A Cathatometer (K) is placed about four feet from the motor and focused on the upper edge of (H) so that the amplitude of vibration can be read to within one ten-thousandth of a foot.

A hand tachometer is sometimes used in conjunction with the strobotac, but all the recorded motor speeds are observed with the strobotac.

With the above equipment, the following items can be varied to thoroughly investigate the action of the non-linear system:

1. Length of cantilever springs. (B)
2. Thickness of beam. (E)
3. Span of beam. (E)
4. Mass and eccentricity of the weights (F) on motor shaft.
5. Motor speed.

LABORATORY PROCEDURE

The equipment is assembled using a given set of variables such as the following: (1) length of cantilever springs, three feet; (2) thickness of beam, 1/4"; (3) span of the beam, four feet; (4) one set of matched eccentric weights. With the motor at rest, the apparatus is so adjusted that the beam B has no initial deflection.

The Strobotac is checked by the vibrating reed at 900 rpm before starting each run.

The Cathatometer is focused on the upper horizontal edge of the piece (H), and the zero reading recorded.

The room is then darkened, and the motor speed increased by successive increments, each speed being obtained by the Strobotac. The Strobotac is then focused on the upper horizontal edge of the piece (H) and set slightly off the rotating speed of the motor so that the sheet metal plate (H) appears to move slowly up and down with the beat frequency resulting. This procedure enables the operator on the Cathatometer to bring the cross hair to the highest position of the horizontal edge of the sheet metal piece (H) with extreme accuracy. The Strobotac is again focused on the motor and the speed rechecked. The motor speed and amplitude reading are recorded and the same procedure is repeated at each successive higher speed.

A run is then made by starting at, say, 1600 rpm and slowly decreasing the speed by stages.

At the end of each run, the zero reading of the Cathatometer is rechecked.

The runs are repeated several times, and those readings in the vicinity of the overlap of the amplitude curve are further checked to minimize the observational error.

Table - A

E= .000175 G = .0226 ω_N= 365 c = 15.11

Speed rpm (obs)	Amplitude ft. (obs)	R (obs)	A (obs)	A (calc)	T (obs)	T (calc)
0	0	0	0	0	1.0	1.0
200	.0001	.548	.0015	.0050	1.43	1.5
265	.0005	.726	.0075	.0150	2.12	2.0
300	.0016	.822	.0249	.0220	3.07	2.9
345	.0051	.945	.0770	.1020	8.84	8.5
360	.0104	.986	.1570	.1850	18.1	15.0
375	.0180	1.028	.2720	.3060	37.1	24.3
385	.0235	1.056	.3550	.3800	41.4	29.3
400	.3000	1.097	.4540	.4710	48.3	36.7
415	.0360	1.137	.5445	.5540	49.9	43.0
425	.0400	1.164	.604	.6050	51.8	46.8
435	.0430	1.191	.650	.6530	52.2	50.1
445	.0465	1.220	.703	.7000	53.0	54.0
450	.0515	1.233	.779	.7210	16.07	54.5
	Decreasing Speed					
980	.0009	2.685	.0136	.0115	0.16	0.2
640	.0012	1.753	.0181	.0180	0.39	0.4
570	.0014	1.561	.0212	.0210	0.68	0.7
505	.0017	1.383	.0257	.0265	1.10	1.00
475	.0021	1.300	.0318	.0315	1.42	1.45
442	.0027	1.210	.0408	.0410	2.22	2.2
415	.0034	1.138	.0574	.0600	3.42	3.5
400	.0057	1.097	.0861	.0800	1.61	5.0

Table - B

E = .0000585 G = .0226 ω_N = 365 c = 15.11

Speed rpm(obs)	Amplitude ft. (obs)	R (obs)	A (obs)	A (calc)	T (obs)	T (calc)
0	0	0	0	0	1.00	1.00
200	.0001	.548	.0015	.0030	1.43	1.40
270	.0004	.740	.0060	.0070	2.20	2.10
325	.0014	.890	.0211	.0030	4.77	5.00
350	.0061	.958	.0920	.0760	10.81	9.20
365	.0109	1.000	.1645	.1850	29.10	25.2
370	.0145	1.014	.2190	.2250	34.60	33.0
375	.0176	1.028	.2660	.2700	39.50	39.0
380	.0204	1.041	.3080	.3100	42.00	42.0
385	.0235	1.055	.3545	.3500	41.65	45.0
390	.0275	1.069	.4150	.3810	30.75	46.5
395	.0320	1.082	.4825	.4135	18.50	47.9
	Decreasing Speed					
595	.0007	1.630	.0106	.0100	0.60	0.60
485	.0010	1.330	.0151	.0180	1.30	1.25
460	.0012	1.260	.0181	.0205	1.70	1.70
440	.0014	1.206	.0211	.0250	2.20	2.25
430	.0016	1.179	.0214	.0285	2.56	2.60
415	.0020	1.138	.0302	.0330	3.39	3.50
410	.0022	1.124	.0332	.0370	3.79	3.90
405	.0024	1.110	.0362	.0400	4.31	4.75
395	.0030	1.082	.0453	.0520	5.38	6.00
390	.0034	1.069	.0513	.0620	7.04	7.10
385	.0038	1.055	.0558	.0760	8.90	10.00

Table - C

E = .0001786 G = .0236 ω_N = 544 c = 15.6

Speed rpm (obs)	Amplitude ft. (obs)	R (obs)	A (obs)	A (calc)
0	0	0	0	0
250	.0001	.460	.0015	.0040
300	.0004	.551	.0062	.0055
395	.0011	.726	.0171	.0160
445	.0025	.836	.0390	.0260
480	.0034	.882	.0530	.0440
530	.0105	.975	.1640	.1530
555	.0175	1.021	.2730	.2900
575	.0250	1.058	.3900	.3800
605	.0328	1.112	.5110	.5100
615	.0360	1.130	.5610	.5430
630	.0382	1.160	.5950	.5950
640	.0405	1.178	.6310	.6300
650	.0435	1.197	.678	.6600
660	.0470	1.214	.733	.6880
	Decreasing Speed			
1010	.0010	1.857	.0156	.0175
900	.0012	1.655	.0187	.0200
800	.0013	1.471	.0203	.0250
700	.0021	1.288	.0328	.0330
630	.0040	1.160	.0624	.0510
590	.0055	1.085	.0858	.0900
580	.0067	1.067	.1046	.1200
570	.0079	1.049	.1232	- -

Table - D

E = .000153 G = .0235 ω_N = 613 c = 14.02

Speed rpm (obs)	Amplitude ft. (obs)	R (obs)	A (obs)	A (calc)
0	0	0	0	0
320	.0001	.522	.0014	.0042
420	.0006	.653	.0084	.0090
450	.0012	.734	.0169	.0190
520	.0026	.848	.0365	.0300
555	.0039	.905	.0547	.0550
595	.0112	.970	.1570	.1360
610	.0148	.995	.2075	.2075
630	.0236	1.029	.3300	.3100
650	.0271	1.060	.3800	.3880
675	.0338	1.101	.4745	.4850
690	.0370	1.127	.5190	.5300
700	.0398	1.141	.5590	.5570
710	.0449	1.159	.6300	.5855
	Decreasing Speed			
1250	.0007	2.040	.0098	.0150
990	.0012	1.615	.0169	.0190
880	.0016	1.436	.0224	.0230
840	.0019	1.370	.0267	.0275
800	.0022	1.305	.0308	.0300
720	.0037	1.175	.0519	.0460
680	.0051	1.110	.0715	.0675
655	.0067	1.069	.0940	.1080
635	.0088	1.047	.1233	- - -

Table E

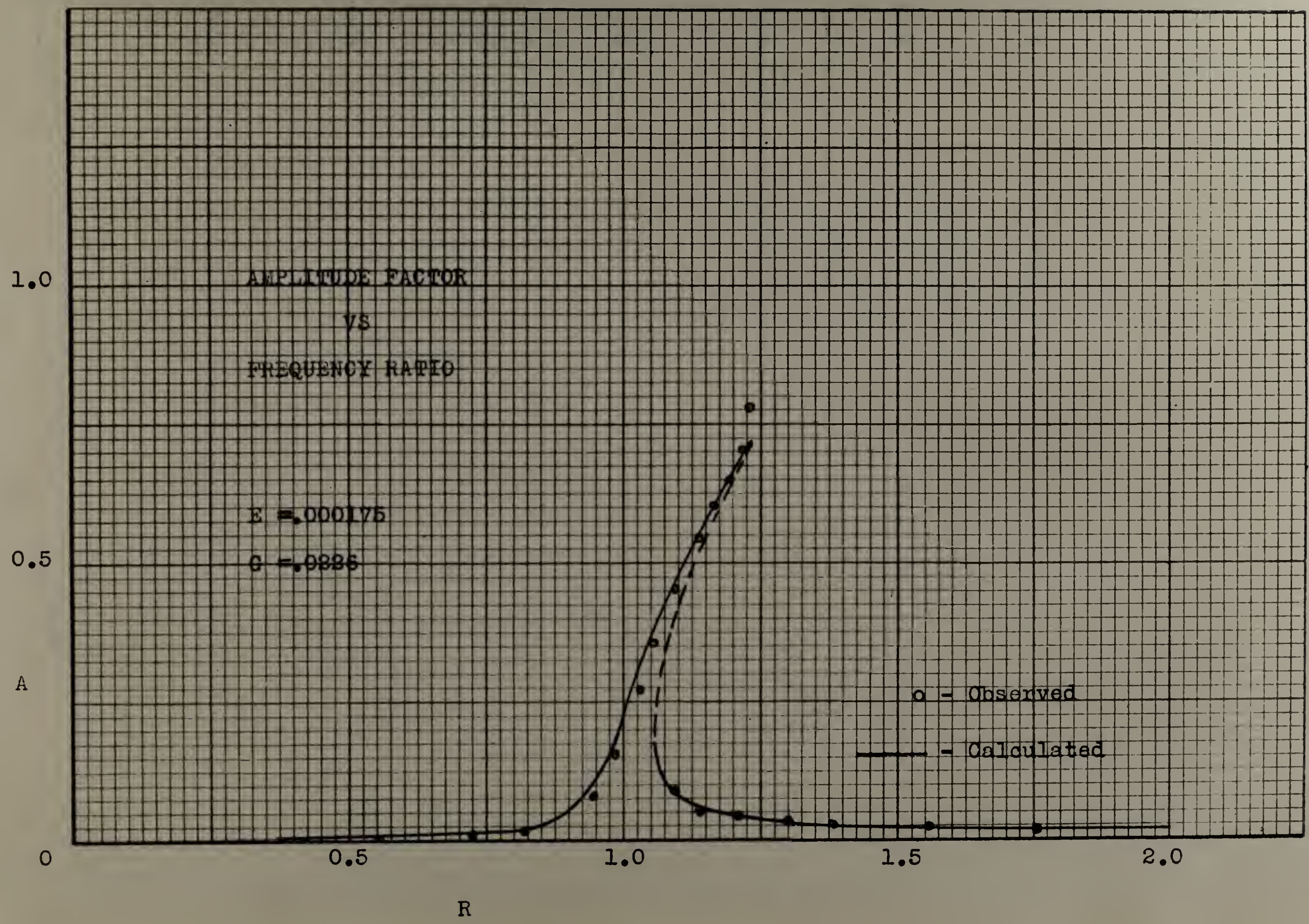
E = .000124 G = .0240 ω_N= 471 c = 12.38

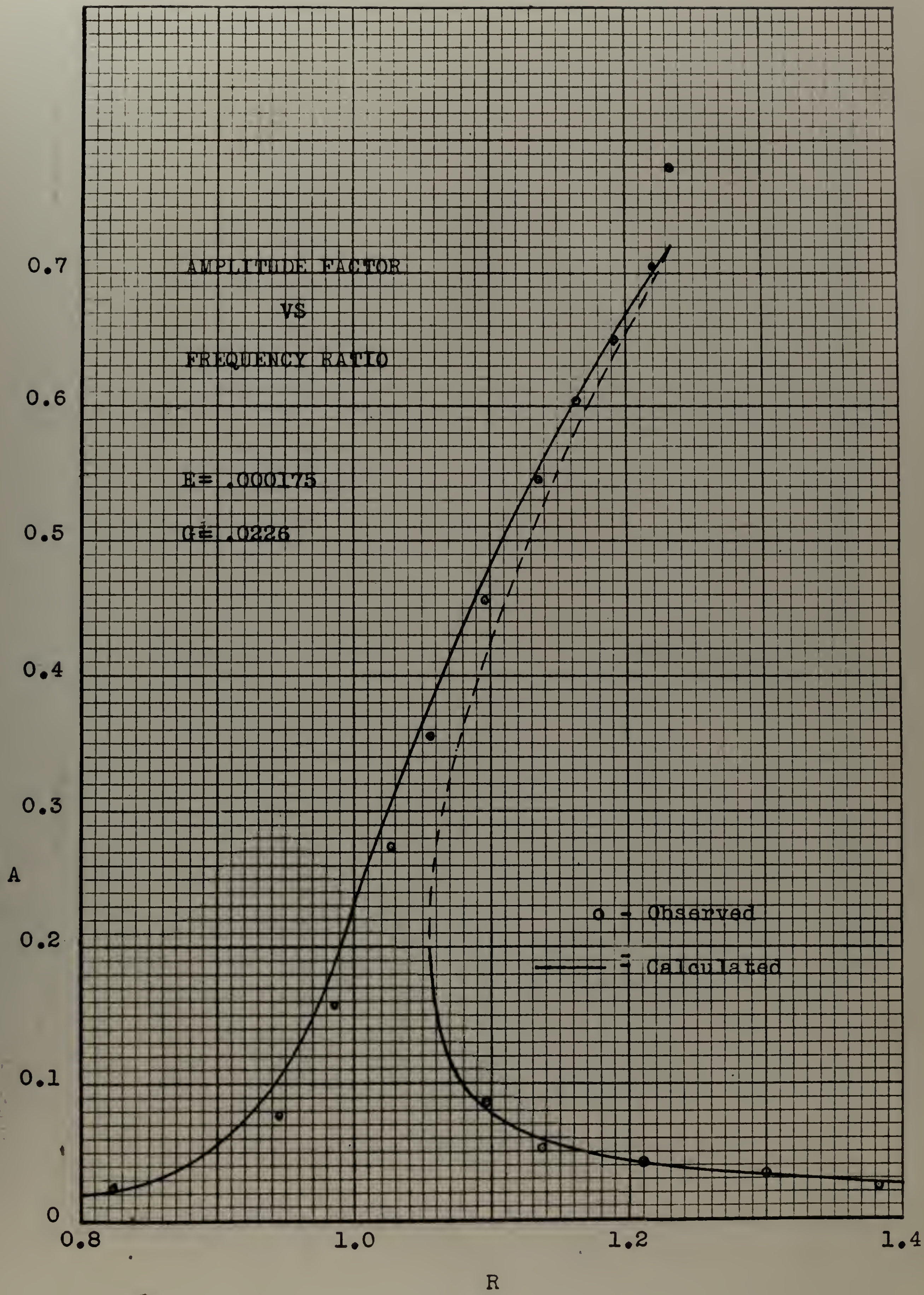
Speed rpm (obs)	Amplitude ft. (obs)	R (obs)	A (obs)	A (calc)
0	0	0	0	0
273	.0001	.580	.0012	.0040
310	.0004	.658	.0050	.0060
370	.0012	.785	.0148	.0190
400	.0029	.850	.0359	.0250
415	.0033	.880	.0408	.0350
425	.0046	.902	.0570	.0500
440	.0062	.934	.0766	.0760
455	.0110	.965	.1360	.1200
470	.0165	1.000	.2040	.2148
490	.0270	1.040	.334	.3500
505	.0320	1.071	.396	.4100
510	.0350	1.083	.433	.4330
515	.0375	1.095	.464	.4600
523	.0410	1.111	.507	.4930
532	.0490	1.130	.606	.5260
	Decreasing Speed			
800	.0004	1.700	.0050	.0150
630	.0018	1.338	.0223	.0250
545	.0034	1.158	.0421	.0420
510	.0051	1.083	.0631	.0790
495	.0068	1.051	.0842	.1300
490	.0076	1.041	.0940	- - -

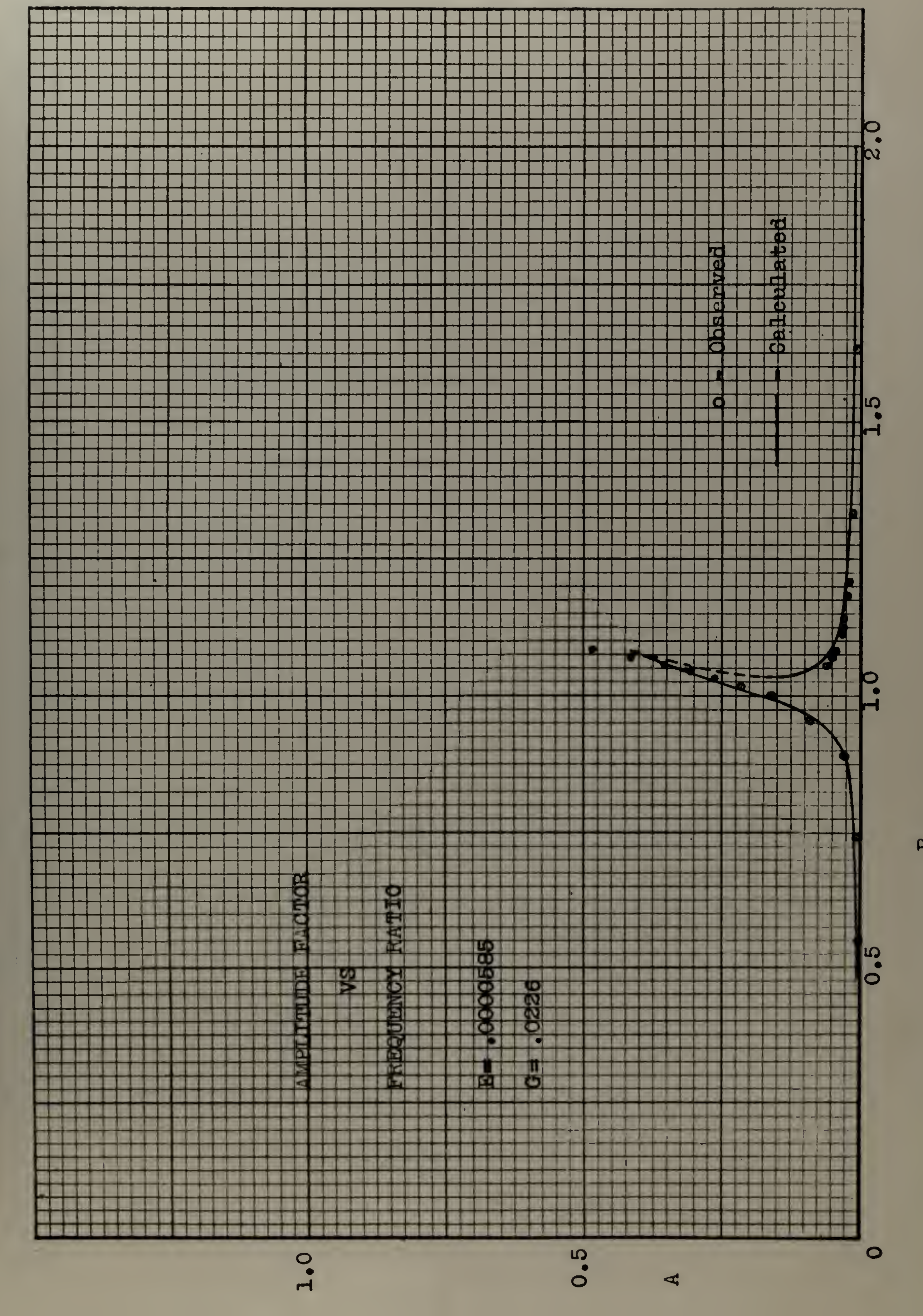
Table F

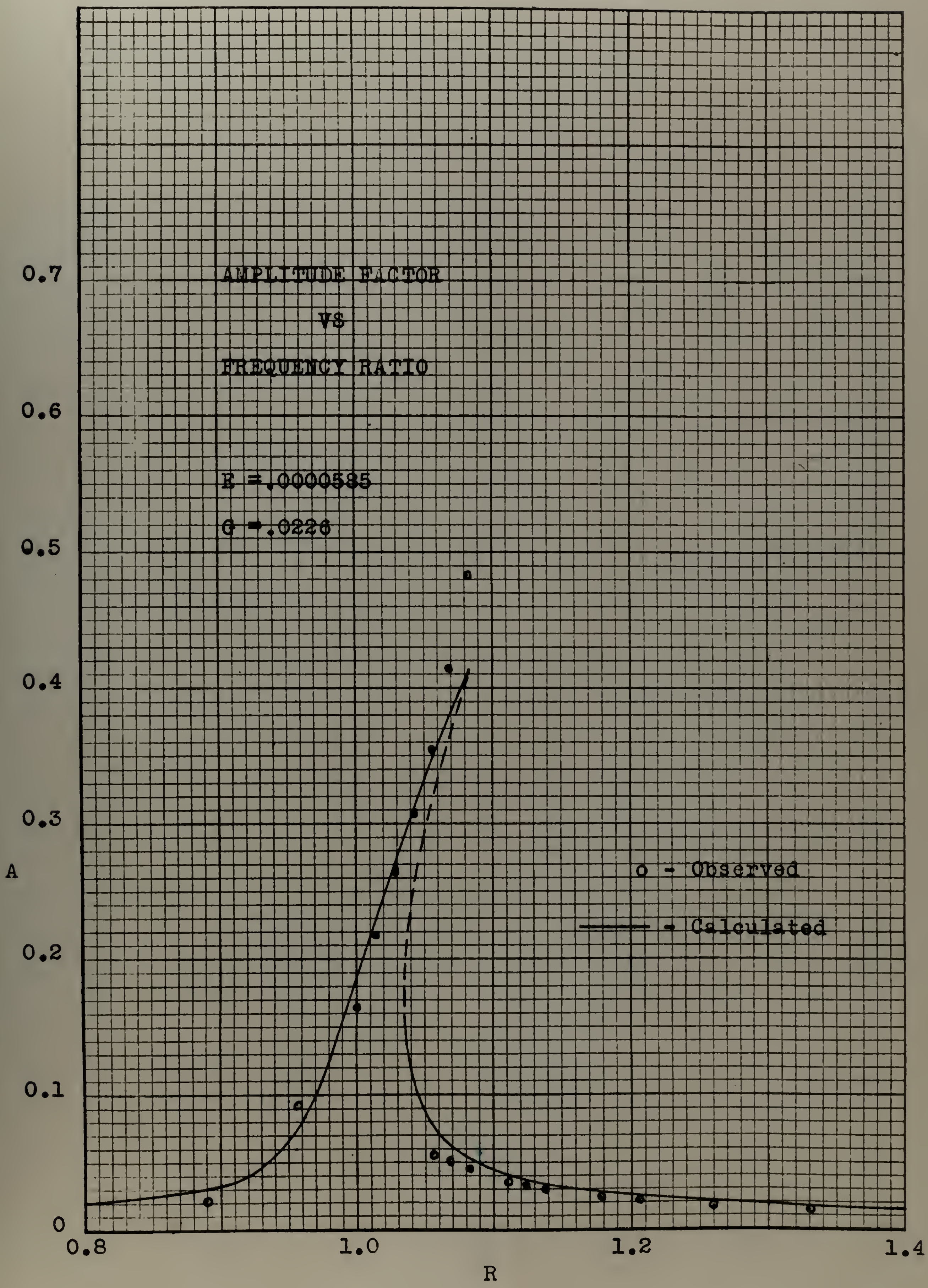
Spring Characteristic Data

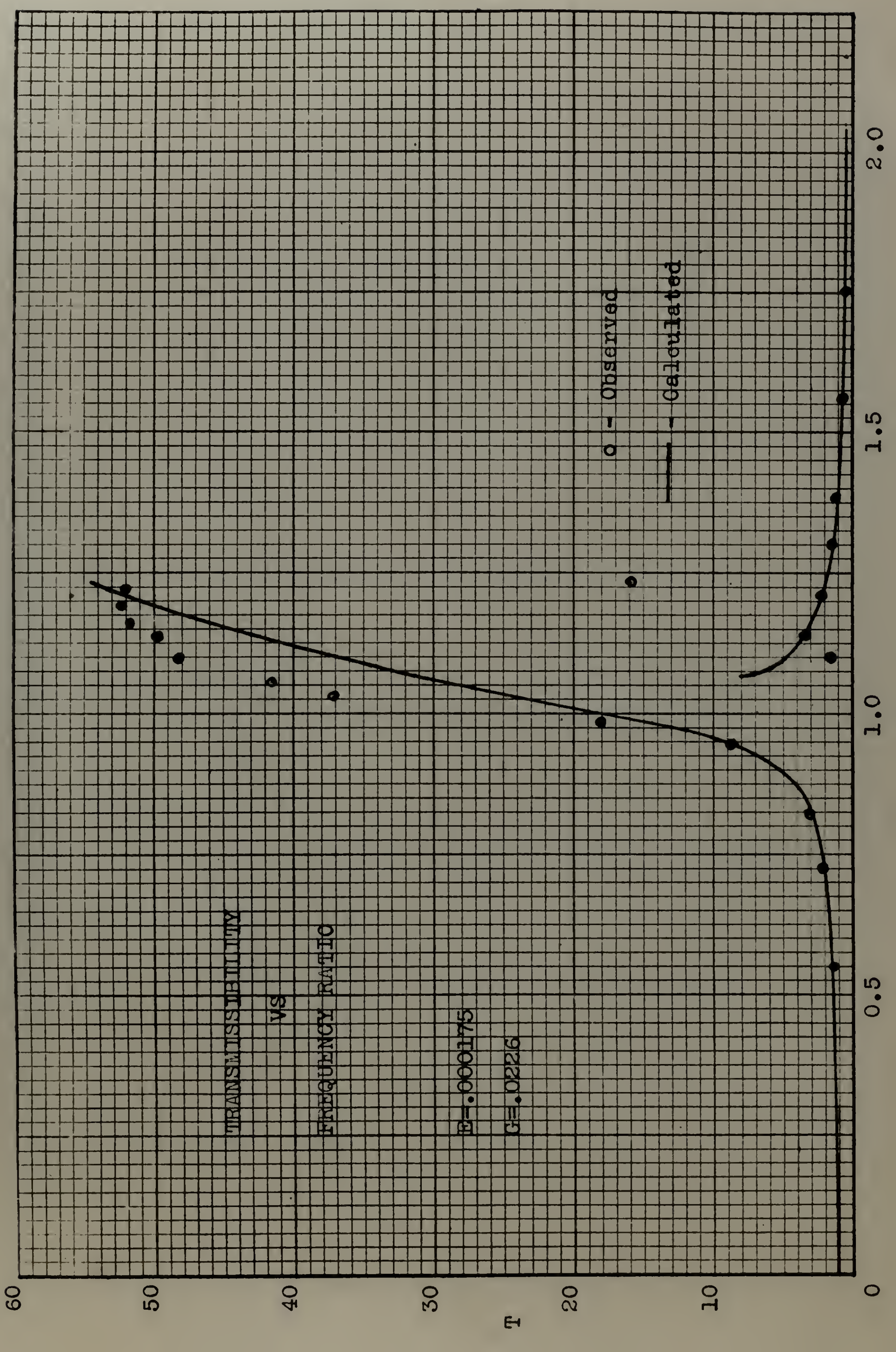
3/8" x 2" x 48" Beam				1/4" x 2" x 48" Beam	
Force lbs.	Deflection feet	Force lbs.	Deflection feet	Force lbs.	Deflection feet
0	0	119.6	.01996	0	0
10	.002	130.8	.02125	5	.00[illegible]5
15.75	.003	140.0	.02246	9.5	.00767
20.0	.004	150.0	.02346	15.0	.01163
25.5	.005	155.5	.02396	19.2	.01398
30.0	.00591	160.0	.02442	24.5	.01683
35.0	.00683	165.5	.02508	29.0	.01892
39.5	.00767	171.4	.02567	33.5	.0208
45.0	.00875	177.0	.02608	39.0	.02288
51.0	.00967	180.5	.02658	45.0	.0248
60.0	.01117	185.4	.02700	50.5	.02646
70.0	.01292	189.5	.02758	54.0	.02746
80.0	.01450	193.5	.02780	59.5	.02875
90.5	.01600	199.5	.02892		
100	.01742	206.0	.02950		
109	.01871	220.5	.03150		

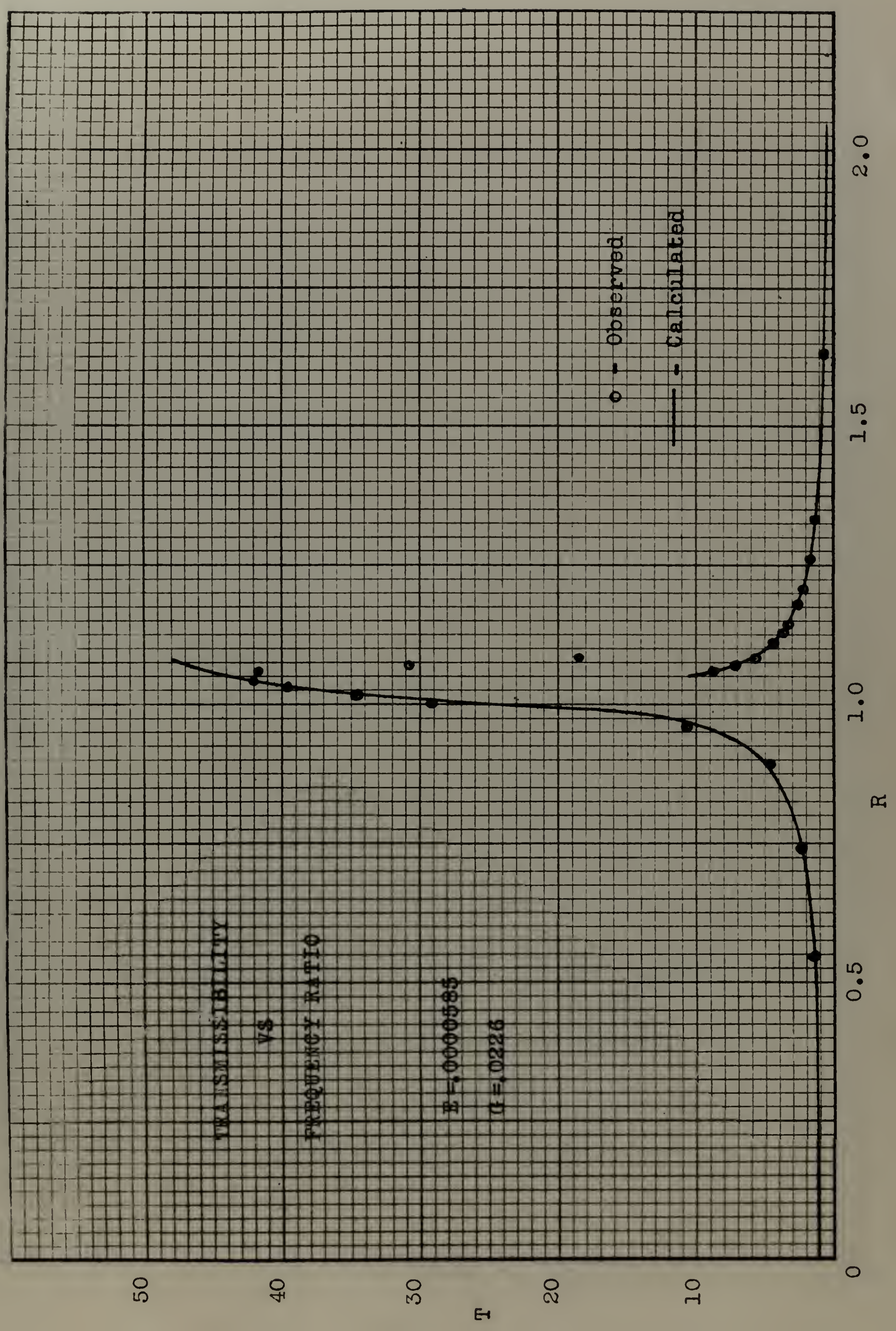

AMPLITUDE FACTOR
VS
FREQUENCY RATIO
E =.000175
G =.0886
o - Observed
- Calculated
1.0
0.5
A
0
0.5
1.0
1.5
2.0
R

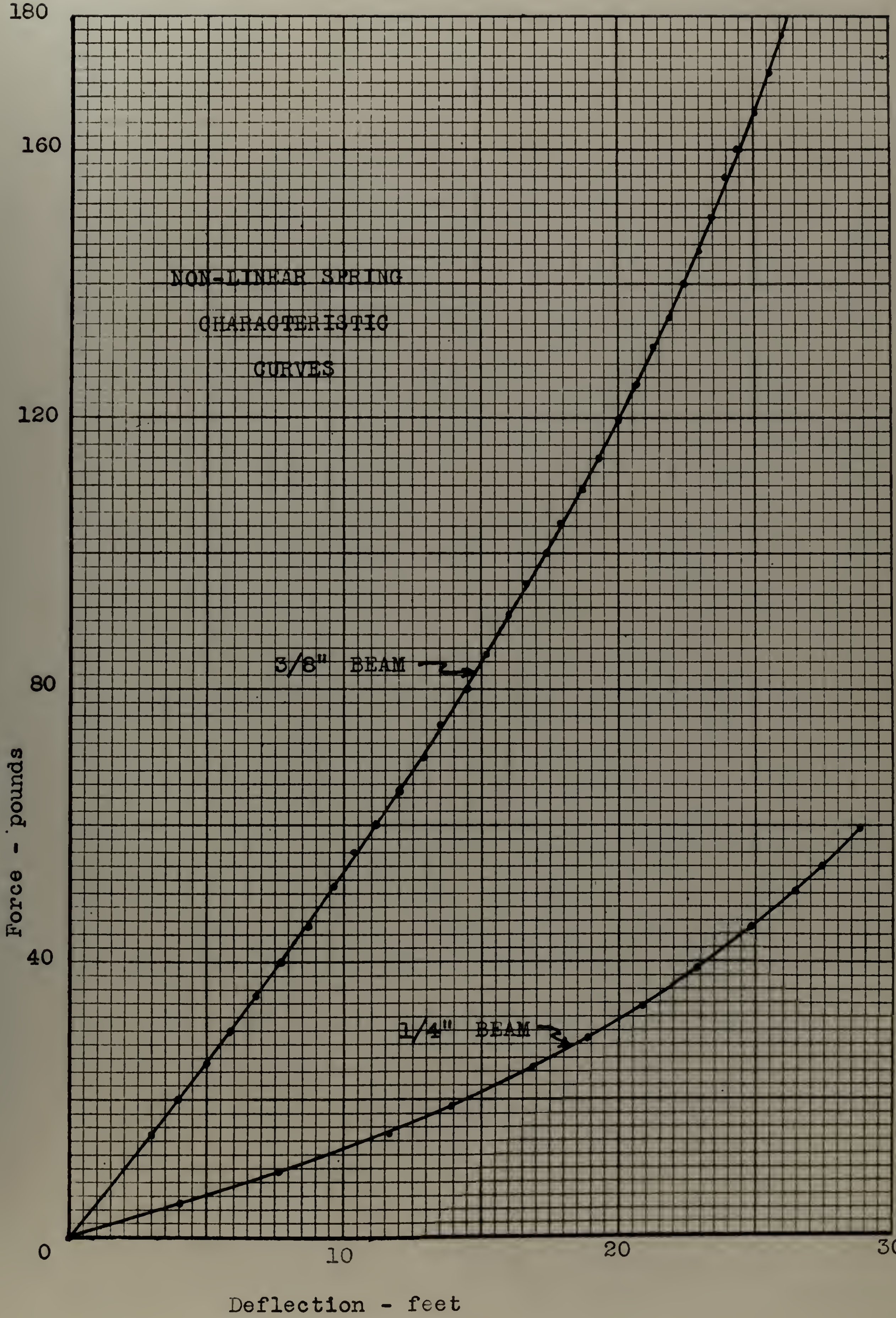

AMPLITUDE FACTOR
VS
FREQUENCY RATIO
E= .000175
G= .0226
o - Observed
—— - Calculated
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
A
0.8
1.0
1.2
1.4
R

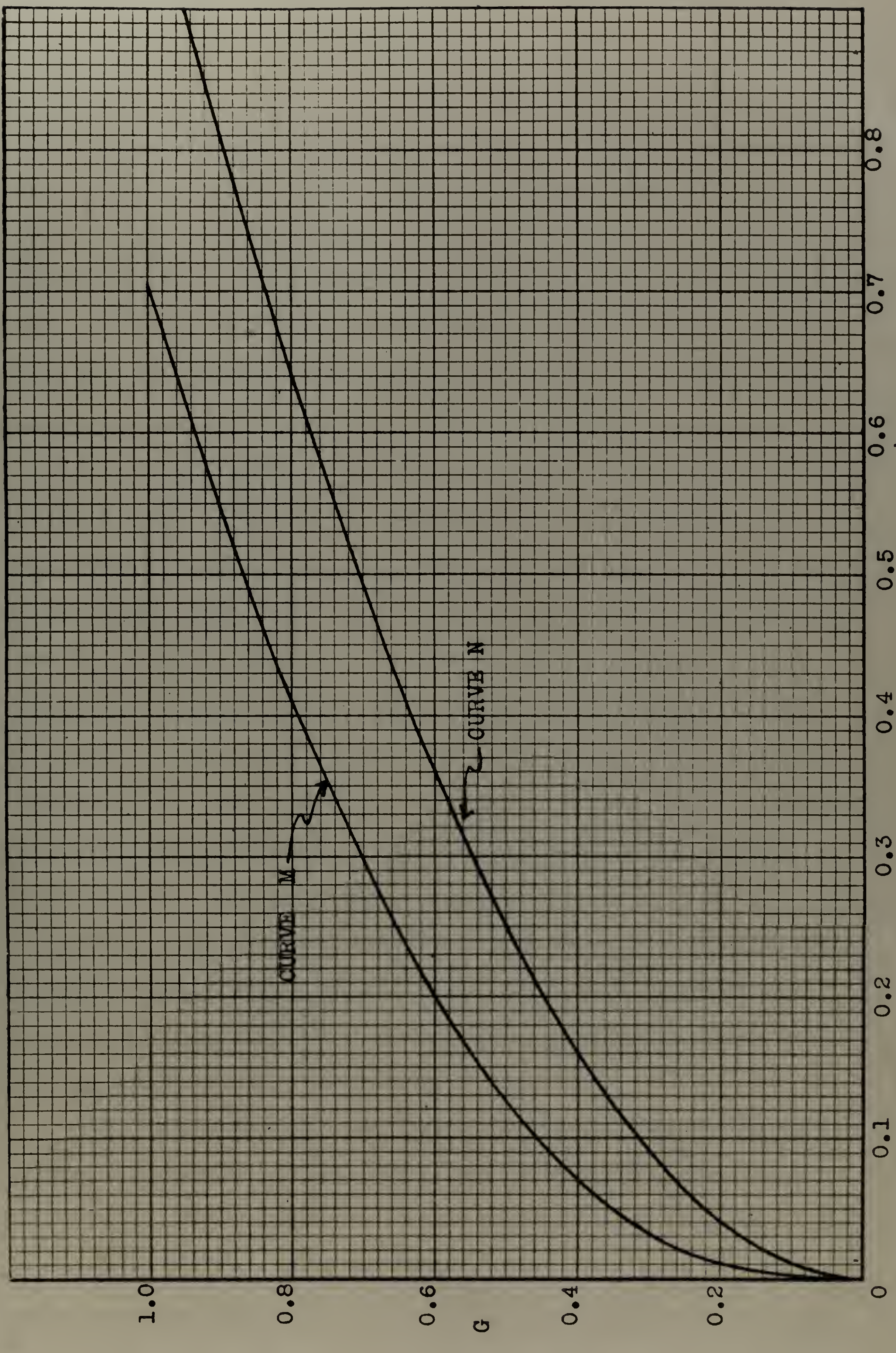

AMPLITUDE FACTOR
VS
FREQUENCY RATIO
E= .0000585
G= .0226
o — Observed
—— Calculated
R
A
0
0.5
1.0
1.5
2.0
0.5
1.0


AMPLITUDE FACTOR
VS
FREQUENCY RATIO
E = .0000585
G = .0226
o - Observed
- Calculated
A
R
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
1.0
1.2
1.4


TRANSMISSIBILITY
VS
FREQUENCY RATIO
E=.000175
G=.0226
o - Observed
— - Calculated
T
R
0
10
20
30
40
50
60
0.5
1.0
1.5
2.0


TRANSMISSIBILITY
VS
FREQUENCY RATIO
E=.0000585
G=.0226
o - Observed
— - Calculated
T
50
40
30
20
10
R
0
0.5
1.0
1.5
2.0


NON-LINEAR SPRING
CHARACTERISTIC
CURVES
3/8" BEAM
1/4" BEAM
180
160
120
80
40
0
10
20
30
Force - pounds
Deflection - feet


CURVE M
CURVE N
G
1.0
0.8
0.6
0.4
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
E

DISCUSSION

The calibration curve for each of the beams with fixed ends, is obtained with the use of a dial indicator and a spring scale. These curves are shown on page 32; the data areon page 25. The cantilever springs are deflected in the same manner as the beam and the calibration curve obtained. With the two calibration curves available, the spring constant for the entire elastic unit is computed.

The coefficient of viscous damping is another constant of the system that must be determined. The record of the free damped vibrations of the elastic unit, without the non-linear beam, indicated the typical logarithmic rate of decay due to viscous damping. A similar record, taken with the non-linear beam attached, gave a diagram in which the rate of decay was not logarithmic. Indeed, this is to be expected even though the damping is viscous. Since no information could be found relative to the theoretical shape of this latter curve, the laboratory records were of no value in determining the coefficient of viscous damping. The Damping Factor G, used in the computations, was obtained from the relationship $G = \sqrt{\frac{E R^2}{R^2 - 1}}$; R being the frequency ratio where the amplitude curve rounds off.

The amplitude data obtained in the laboratory closely checked the curves plotted from the derived formula. The curves shown on pages 26 - 29 are plotted from representative experimental data. Enlargements of the important regions of the curves on pages 26 and 28 are shown on pages 27 and 29,

DISCUSSION

The calibration curve for each of the beams with fixed ends, is obtained with the use of a dial indicator and a spring scale. These curves are shown on page 20; the data are on page 48. The deflections are reflected in the same manner as the beam and the calibration curve obtained. With the two calibration curves available, the spring constant for the entire elastic unit is computed.

The coefficient of viscous damping is another parameter of the system that must be determined. The record of the free damped vibrations of the elastic unit, without the non-linear term, indicated the typical logarithmic rate of decay due to viscous damping. A similar record, taken with the non-linear term attached, gave a diagram in which the rate of decay was not logarithmic. Indeed, this is to be expected even though the damping is viscous. Since no information could be found relating to the theoretical shape of this latter curve, the laboratory records were of no value in determining the coefficient of viscous damping. The damping factor C, used in the computations, was obtained from the relationship $C = \sqrt{\frac{E R^2}{R^2-1}}$; R being the frequency ratio where the amplitude curve rounds off.

The resonance data obtained in the laboratory closely checked the curves plotted from the derived formulas. The curves shown on pages 25 – 28 are plotted from representative experimental data. Enlargements of the important regions of the curves on pages 25 and 26 are shown on pages 27 and 28.

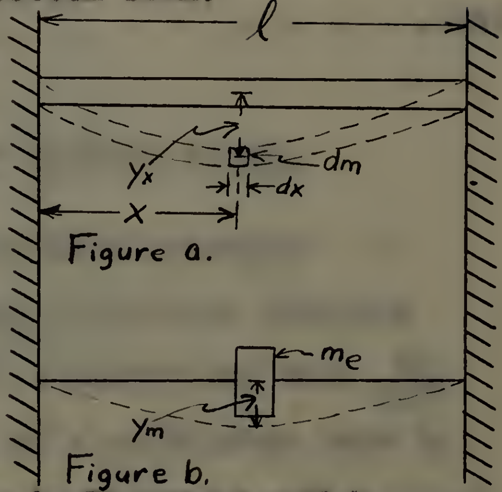
respectively. The error at frequency ratios less than unity averaged 21 per cent. The error at frequency ratios greater than unity for the J branch of the curve (Figure 3, page 8), averaged 3.6 per cent. The error for the L branch of the curve averaged 12 per cent. The large errors occurred where the amplitude is low. This is to be expected since a small error (.0001') in reading the amplitude may cause the computed value to differ from the theoretical value by as much as 50 per cent. Fortunately, the error in the measured values of amplitude was least over the most important portion of the curve. For the assumptions made in the derivation of the formula, the small amount of error (3.6 per cent) in the experimental values justifies the statement that the derived formula is sufficiently accurate to predict the amplitude of a vibrating system with a non-linear spring.

The Transmissibility curves on pages 30 and 31 show a large difference between the predicted and experimental values. This discrepancy was investigated and it was found that a small error in the amplitude produced a relatively large error in the Transmissibility. Experimental values of amplitude that lie below the predicted amplitude curve will give larger values of Transmissibility than the predicted values.

SAMPLE CALCULATIONS

Total Mass Involved in the Vibration;

(a) Effect of the mass of the bottom beam.



Let m = total mass of beam

m_e = equivalent mass considered concentrated at the mid-point of massless beam.

Equating the Kinetic Energies of the two systems in Figures a and b

$$1/2 \; m_e \; \dot{y}^2_{m(Max)} = 1/2 \int_{x=0}^{x=\ell} dm \; \dot{y}^2_{x(Max)} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (1)$$

Assuming the first mode of vibration and the dynamical deflection curve to be the same as the static deflection curve,

$$y_x = \frac{W\ x^2}{24\ EI} (l - x)^2 \sin \omega_f t \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (2)$$

$$\dot{y}_x = \frac{W\ x^2 \omega_f}{24\ EI} (l-x)^2 \cos \omega_f t \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (3)$$

$$\dot{y}_{x(Max)} = \dot{y}_{x_{\omega_f t=0}} = \omega_f \left[\frac{W\ x^2}{24\ EI} (l-x)^2 \right] \; . \; . \; . \; . \; . \; . \; . \; . \; (4)$$

$$\dot{y}^2_{x(Max)} = \omega^2 \left[\frac{W\ x^2}{24\ EI} (l-x)^2 \right]^2 \; . \; . \; . \; . \; . \; . \; . \; . \; (5)$$

When $x = \frac{l}{2}$

$$\dot{y}^2_{m(Max)} = \omega^2 \left[\frac{W\ l^3}{384\ EI} \right]^2 \; . \; . \; . \; . \; . \; . \; . \; . \; (6)$$

Referring to Figure (a)

$$dm = \frac{m}{l} dx \; . \; . \; . \; . \; . \; . \; . \; . \; (7)$$

Substituting equations 5 6 and 7 into equation 1

$$1/2 \; m_e \omega^2 \left[\frac{W\ l^3}{384\ E\ I} \right]^2 = 1/2 \int_{x=0}^{x=\ell} \omega^2 \left[\frac{W\ x^2}{24\ EI} (l-x)^2 \right]^2 \frac{m}{l} \; dx \; . \; . \; . \; . \; (8)$$

Equation 8 when simplified becomes

$$m_e = m \frac{(384)^2}{l^9 (24)^2} \int_{x=0}^{x=l} \left[x^2 (1-x)^2 \right]^2 dx \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (9)$$

The integral when evaluated $= \frac{l^9}{630}$; so that

$$m_e = 0.4625 \; m \qquad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (10)$$

For a beam 1/4"x2" x 48"

$$m_e = \frac{48x2x.283x.4625}{4 \times 32.2} = \underline{0.09755 \text{ slugs.}}$$

(b) Effect of the mass of the cantilever springs.

By the above method, Timoshenko (Vibration Problems in Engineering, pages 57 - 58) demonstrates that $\frac{33}{140}$ of the total distributed mass of a cantilever beam is considered concentrated at the free end of a "massless" beam.

For two 3-foot springs (1/2" x 5" x 36")

$$m = \frac{33x36x5x2x0.283}{140x2x40x32.2} = \underline{0.373 \text{ slugs.}}$$

(c) Weight of the motor, vertical channel, etc.

$$m = \frac{57}{32.2} = \underline{1.77 \text{ slugs.}}$$

Total Effective Mass $M = \underline{2.241 \text{ slugs.}}$

Spring Force

(a) From the characteristic curve of the 1/4" x 2"x 48" beam shown on page 32, the equation for the spring force is;

$$F = 1.151 \times 10^3 \, y + 1.084 \times 10^6 \, y^3 \qquad F = \text{pounds}, \quad y = \text{feet}$$

(b) For the spring force of the two cantilever springs, 3 feet long,

$$F = k\,y \qquad \text{where } k = \frac{3\,EI}{l^3}$$

$$k = \frac{3x30x10 \;\; x5x12}{(36)^3 x8 \times 12} = 1205 \text{ lbs/foot.}$$

or, For two springs, $F = 2410\ y$.

By the method of superposition the total spring force is;

$F = 3.561 \times 10^{3}\ y + 1.084 \times 10^{6}\ y^{3}$

where $k = 3.561 \times 10$ lbs/foot

$k_1 = 1.084 \times 10$ lbs/foot^3

Natural Frequency

$$\omega_N = \sqrt{\frac{k}{M}} = \sqrt{\frac{3.561 \times 10^3}{2.241}} = 38.20 \text{ rad/sec, or } 365 \text{ rpm.}$$

Large Eccentric Weights

$m = 0.0353$ slugs

$e = 0.0556$ feet

$me = 0.00196$ slug - feet.

Exciting Force E (dimensionless term).

$$E = \frac{3\ k_1 m^2 e^2}{4\ k\ M^2} = \frac{3 \times 1.084 \times 10^6 \times \overline{0.00196}^{\,2}}{4 \times 3.561 \times 10^3 \times \overline{2.241}^{\,2}} = \underline{0.000175}$$

Damping Factor G

Using the relationship $G = \sqrt{\frac{ER^2}{R^2-1}}$, where R is the frequency ratio at the point where the amplitude curve rounds off. ($R = 1.233$)

$$G = \sqrt{\frac{0.000175 \times (1.233)^2}{(1.233)^2 - 1}} = \underline{0.0226}$$

Amplitude Factor A

From equation 14, page 8

$$A^2\left[\left(1-R^2+A^2\right)^2 + G^2R^2\right] = E\ R^4$$

or,

$$A^6 + 2A^4\left(1-R^2\right) + A^2\left[\left(1-R^2\right)^2 + G^2R^2\right] - E\ R^4 = 0$$

For the particular case where

$R = 1.10$ (assumed)

$E = 0.000175$ (calculated)

$G = 0.0226$ (Calculated)

then

$$A^6 - 0.42\,A^4 + 0.04472\,A^2 - 0.0002566 = 0$$

The three roots are

$A^2 = 0.00605$ or $A = 0.0779$

$A^2 = 0.2322$ Or $A = 0.482$

$A^2 = 0.1818$ or $A = 0.426$

<u>Transmissibility</u>

$$T = \sqrt{\frac{(1+A^2)^2 + G^2R^2}{(1-R^2+A^2)^2 + G^2R^2}}$$

For the assumed case in which

$R = 1.10$ (assumed)

$G = 0.0226$ (calculated)

$A = 0.2322$ (calculated)

$$T = \sqrt{\frac{(1.2322)^2 + 0.0006175}{(0.0222)^2 + 0.0006175}} = \underline{37.}$$

APPENDIX

II References

1. E. V. APPLETON, The Motion of a Vibration Galvanometer, Phil. Mag., Series 6, Vol. 47 (1924) page 609.

2. S. TIMOSHENKO, Vibration Problems in Engineering, (1937) pages 114 - 150.

3. J. P. DEN HARTOG, Mechanical Vibrations, pages 348 - 358.